On locations and properties of the multicritical point of Gaussian and $\pm J$ Ising spin glasses

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We use transfer-matrix and finite-size scaling methods to investigate the location and properties of the multicritical point of two-dimensional Ising spin glasses on square, triangular and honeycomb lattices, with both binary and Gaussian disorder distributions. For square and triangular lattices with binary disorder, the estimated position of the multicritical point is in numerical agreement with recent conjectures regarding its exact location. For the remaining four cases, our results indicate disagreement with the respective versions of the conjecture, though by very small amounts, never exceeding 0.2%. Our results for: (i) the correlation-length exponent ν governing the ferro-paramagnetic transition; (ii) the critical domain-wall energy amplitude η; (iii) the conformal anomaly c; (iv) the finite-size susceptibility exponent γ/ν; and (v) the set of multifractal exponents $\{\eta_k\}$ associated to the moments of the probability distribution of spin-spin correlation functions at the multicritical point, are consistent with universality as regards lattice structure and disorder distribution, and in good agreement with existing estimates.

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I. INTRODUCTION

In this paper we investigate quenched random-bond Ising spin - 1/2 models on regular two-dimensional lattices, namely square [SQ], triangular [T], and honeycomb [HC]. For suitably low concentrations of antiferromagnetic bonds, it is known that such systems exhibit ferromagnetic order at low temperatures. We consider only nearest-neighbor couplings J_{ij}, with strengths extracted from identical, independent probability distribution functions (PDFs). We specialize to the following two forms for the latter:

$$P(J_{ij}) = p\,\delta(J_{ij}-J_0) + (1-p)\,\delta(J_{ij}+J_0) \qquad (\pm J)\ ;$$
$$P(J_{ij}) = \frac{1}{\sqrt{2\pi}\sigma}\,\exp\left(-\frac{(J_{ij}-J_0)^2}{2\sigma^2}\right) \qquad (\text{Gaussian})\ . \quad (1)$$

Our units are such that $J_0 \equiv 1$ in the former case, and $\sigma \equiv 1$ in the latter. A critical line on the $T-p$ ($\pm J$), or $T-J_0$ (Gaussian), plane separates paramagnetic and ferromagnetic phases; a spin-glass phase for comparable amounts of plus and minus couplings is absent here, on account of the systems under consideration being two-dimensional.

For general space dimensionality $d \geq 2$ there is a second line of interest on the temperature-disorder plane, along which the internal energy has a simple analytic expression, and several exact results have been derived, known as the *Nishimori line* (NL)[1,2]. The shape of the NL is known exactly, and given by

$$e^{-2/T} = \frac{1-p}{p} \qquad (p > \frac{1}{2}) \qquad (\pm J)\ ;$$
$$T = \frac{1}{J_0} \qquad (\text{Gaussian})\ . \qquad (2)$$

The intersection of the ferro-paramagnetic boundary with the NL is a multicritical point[3], the *Nishimori point* (NP). A conjecture regarding the possibly exact location of the NP has been put forward, which invokes the effects of duality and gauge symmetry arguments on the replicated partition function of quenched random Z_q models[4,5,6]. With further extensions to consider non self-dual lattices[7,8], numerically exact predictions have been produced, for the Z_2 (Ising) model, for all lattices and interaction distributions considered here. Versions of the conjecture adapted for hierarchical lattices have been considered as well[9].

Locations of the NP predicted by the conjecture generally agree very well with results obtained by other means. However, the remaining discrepancies provide compelling evidence that, at least in some cases, the conjecture may not be exact. First, on the SQ lattice, several very accurate numerical estimates for the $\pm J$ coupling distribution place the conjectured location[4], $p_c = 0.889972\ldots$, outside the corresponding error bars (though it differs from the central value typically by less than 0.1%). One has: $p_c = 0.8906(2)$ (Refs. 10,11), $0.8907(2)$ (Ref. 12), $0.89081(7)$ (Ref. 13). For a Gaussian distribution, the conjecture gives[4,5] $J_{0c} = 1.021770$, while Ref. 11 finds $J_{0c} = 1.02098(4)$. Second, it has been shown that the exact renormalization-group solution for three pairs of mutually dual hierarchical lattices disagrees with the pertinent form of the conjecture, by up to 2%[9].

Very recently, these issues have been addressed via the proposal of an improved conjecture, first applied to hierarchical lattices[14], and later extended to regular ones[15]. Broadly, this corresponds to considering duality properties applied to a (usually small) cluster of sites on the lattice under examination[14,15], as opposed to the original conjecture which deals only with the partition function of a single bond (the principal Boltzmann factor)[4]. The improved conjecture predicts the location of the NP to be well within the error bars of recent numerical work

for the SQ $\pm J$ case: an average over four slightly differing implementations gives $p_c = 0.89079(6)$, though so far disagreement persists for the Gaussian distribution, as the improved estimate is $J_{0c} = 1.021564$[15]. For hierarchical lattices, the gap between conjecture and exact renormalization-group solutions has essentially been bridged by the new approach[14].

Existing numerical results for T and HC lattices ($\pm J$ distribution only)[7,16] broadly agree with an early form of the original conjecture, applicable to pairs of dual lattices[7]: with the binary entropy

$$H(p) \equiv -p \log_2 p - (1-p) \log_2 (1-p) , \qquad (3)$$

it is predicted that, for a pair of mutually-dual lattices 1 and 2,

$$H_{12} \equiv H(p_{1c}) + H(p_{2c}) = 1 . \qquad (4)$$

Ref. 7 finds $p_c = 0.835(5)$ and $0.930(5)$, respectively for T and HC, which implies $0.981 < H_{12} < 1.042$; these estimates were refined in Ref. 16 to $p_c = 0.8355(5)$ [T] and $0.9325(5)$ [HC], giving $H_{12} = 1.002(3)$.

Further developments[8,15] enabled the production of pairs of individual predictions (always obeying Eq. (4), with a suitably-adapted form of Eq. (3) for the Gaussian case). In the framework of the original conjecture, these are: $p_c = 0.835806$ [T] and 0.932704 [HC] ($\pm J$)[8]; $J_{0c} = 0.798174$ [T] and 1.270615 [HC] (Gaussian)[15]. For the improved conjecture ($\pm J$ only), two slightly differing implementations give the pairs: $p_c = 0.835956$ [T] and 0.932611 [HC]; $p_c = 0.835985$ [T] and 0.932593 [HC][15].

Here, we numerically estimate the location and critical properties of the NP on the T and HC lattices. For the $\pm J$ case, we refine the results given in Ref. 16, checking our data against the more stringent predictions of Refs. 8, 15; for the Gaussian distribution, we are not aware of any existing results, apart from those given in Ref. 15 for the conjectured location of the NP. For completeness, and to provide consistency checks of our methods, we revisit the SQ lattice problem, investigating both distributions.

We apply numerical transfer-matrix (TM) methods to the spin–1/2 random-bond Ising model, on strips of SQ, T, and HC lattices, of widths $4 \leq N \leq 14$ sites (SQ), $4 \leq N \leq 13$ sites (T) and $4 \leq N \leq 14$ sites (even values only, HC). We take long strips, usually of length $M = 2 \times 10^6$ columns (pairs of columns for HC, because two iterations of the TM are needed to restore periodicity). For each of the quantities evaluated here, averages $\langle \mathcal{Q} \rangle$ are taken over, and fluctuations $\langle \Delta \mathcal{Q} \rangle_{\rm rms}$ calculated among, N_s independently-generated samples, each of length M. As discussed extensively elsewhere[17], the sample-to-sample fluctuations $\langle \Delta \mathcal{Q} \rangle_{\rm rms}$ vary with $M^{-1/2}$, and are essentially N_s–independent, provided that N_s is not very small. The averaged values $\langle \mathcal{Q} \rangle$ themselves still fluctuate slightly upon varying N_s, but the corresponding fluctuations $\Delta \langle \mathcal{Q} \rangle$ die down with increasing N_s. We found that, for M as above, making $N_s = 10$ already gives $\Delta \langle \mathcal{Q} \rangle / \langle \Delta \mathcal{Q} \rangle_{\rm rms} \lesssim 0.1$, thus this constitutes an adequate compromise between accuracy and CPU time expense. Typical upper bounds for $\langle \Delta \mathcal{Q} \rangle_{\rm rms} / \langle \mathcal{Q} \rangle$ are 10^{-4} for free energies, 10^{-3} for domain-wall energies (see Section II below for definitions).

Table I: Intervals Δp, ΔJ_0 scanned along the NL in our calculations, for lattices and coupling distributions [binary ($\pm J$) and Gaussian (G)] specified in column 1. $\mathcal{N}_p$ gives total number of pairs (p, N) or (J_0, N) at which quantities of interest were calculated. See text.

Type	Δp, ΔJ_0	$\mathcal{N}_p$
SQ, $\pm J$	0.8868 – 0.8948	123
SQ, G	1.00125 – 1.03875	140
T, $\pm J$	0.830 – 0.842	126
HC, $\pm J$	0.9266 – 0.9386	86
T, G	0.7794– 0.8169	134
HC, G	1.254 – 1.287	94

We scanned suitable intervals of p or J_0 along the NL, spanning conjectured and (when available) numerically-calculated positions of the NP, as shown in Table I. For a given lattice and interaction distribution, we took samples at $N_p = N_p(N)$ equally-spaced positions for each lattice width N, generally starting with $N_p \geq 18$ for small N, and decreasing to $N_p = 9$ for $N \geq 8$, giving the totals denoted by $\mathcal{N}_p = \sum_N N_p(N)$ in Table I.

The Mersenne Twister random-number generator[18] was used in all calculations described below. In all calculations pertaining to the $\pm J$ disorder distribution, a canonical ensemble was used, i.e. for a given nominal concentration p of positive bonds, these were drawn from a reservoir initially containing $\alpha_i pNM$ units ($\alpha_i = 2, 3, 3$ respectively for $i =$ SQ,T, HC). This way, one ensures that fluctuations in calculated quantities are considerably smaller than if a grand-canonical implementation were used[11,19].

In Sec. II, domain-wall energies are computed, and their finite-size scaling allows us to estimate both the location of the NP along the NL, and the correlation-length index, $y_t \equiv 1/\nu$ which governs the spread of ferromagnetic correlations upon crossing the ferro-paramagnetic phase boundary. The conformal anomaly, or central charge, is evaluated in Sec. III. In Sec. IV, uniform susceptibilites are calculated, and the associated exponent ratio γ/ν is evaluated (for Gaussian coupling distributions only). In Sec. V, we specialize to T and HC lattices, with Gaussian disorder distributions, and investigate the moments of assorted orders of the probability distributions of spin-spin correlation functions. Finally, in Sec. VI, concluding remarks are made.

II. DOMAIN-WALL SCALING

For a strip of width L, in lattice parameter units, of a two-dimensional spin system, the domain-wall free energy

σ_L is the free energy per unit length, in units of T, of a seam along the full length of the strip. For Ising spins, $\sigma_L = f_L^A - f_L^P$, with f_L^P (f_L^A) being the corresponding free energy for a strip with periodic (antiperiodic) boundary conditions across. Within a TM description of disordered systems, $\sigma_L = -\ln(\Lambda_0^A/\Lambda_0^P)$ where $\ln\Lambda_0^P$, $\ln\Lambda_0^A$ are the largest Lyapunov exponents of the TM, respectively with periodic and antiperiodic boundary conditions across.

The duality between correlation length ξ and interface tension σ is well-established[20] for pure two-dimensional systems, and carries over to disordered cases. In a finite-size scaling (FSS) context[21], this means that σ_L must scale with $1/L$ at criticality, a fact which has been used in previous studies of disordered systems[22], including investigations of the NP[10,11,16]. From conformal invariance[23] one has, at the critical point:

$$L\,\sigma_L = \pi\eta\ , \tag{5}$$

where, for pure systems, η is the same exponent which characterizes the decay of spin-spin correlations. In the presence of disorder, however, the scaling indices of the disorder correlator (i.e., the interfacial tension) differ from those of its dual, the order correlator (namely, spin-spin correlations)[24]. Nevertheless, the constraints of conformal invariance still hold, thus the amplitude of the domain wall energy remains a *bona fide* universal quantity[24]. For the NP, recent estimates on the SQ lattice ($\pm J$ couplings) give $\eta = 0.691(2)$[10,12,24].

We have calculated Λ_0^P, Λ_0^A for strips of SQ, T and HC lattices. Recalling that both L in Eq. (5) and the correlation length ξ (of which the surface tension is the dual) are actual physical distances in lattice parameter units[25,26,27,28], one finds (see Ref. 16) that, in terms of the number of sites N across the strip, the appropriate expressions for the scaled domain-wall energy are of the form: $\eta_N = \eta_N(T,z) = \zeta_i\,(N/\pi)\big(\ln\Lambda_0^P(T,z) - \ln\Lambda_0^A(T,z)\big)$, with $\zeta_i = 1,\ 2/\sqrt{3},\ \sqrt{3}/2$ respectively for $i =$ SQ, T, HC, and where $z = p$ ($\pm J$) or J_0 (Gaussian). At (T_c, z_c) one must have $\lim_{N\to\infty}\eta_N = \eta$, the latter being a universal quantity.

Close to the multicritical NP, the scaling directions are respectively the NL itself, and the temperature axis[3,13,29]. Therefore (neglecting corrections to scaling), along the NL the single relevant variable corresponds to $z - z_c$.

According to finite-size scaling[21], the curves of scaled domain-wall energy calculated for different values of N, T, z along the NL should then coincide when plotted against $x \equiv N^{1/\nu}\,(z - z_c)$.

Bearing in mind that corrections to scaling may be present[11,13], we allow for their effect from the start. Thus, we write[13]:

$$\eta_N = f[N^{1/\nu}\,(z-z_c)] + N^{-\omega} g[N^{1/\nu}\,(z-z_c)]\ , \tag{6}$$

where $\omega > 0$ is the exponent associated to the leading irrelevant operator. Close enough to the NP the scaling functions in Eq. (6) should be amenable to Taylor expansions. One has:

$$\eta_N = \eta + \sum_{j=1}^{j_m} a_j\,(z-z_c)^j\,N^{j/\nu} + N^{-\omega}\sum_{k=0}^{k_m} b_k\,(z-z_c)^k\,N^{k/\nu}\ . \tag{7}$$

We adjusted our TM data to Eq. (7), by means of multiparametric nonlinear least-squares fits. The goodness of fit was measured by the (weighted) χ^2 per degree of freedom ($\chi^2_{\rm d.o.f.}$). We tested several assumptions on k_m, j_m, ω, via their effect on: (i) the resulting $\chi^2_{\rm d.o.f.}$, (ii) the stability of the final estimates for z_c, η, $1/\nu$, and (iii) the broad compatibility of estimates for η and $1/\nu$ with existing results for assorted two-dimensional lattices and coupling distributions (under the reasonable assumption of universality, which is however provisional, and must be weighted against the bulk of available evidence).

We found that:
(1) a parabolic form, $j_m = 2$, is adequate for the description of the broad features of data, similarly to conclusions from the Monte-Carlo study of Ref. 13;
(2) Neglecting corrections to scaling (all $b_k \equiv 0$) generally gave a $\chi^2_{\rm d.o.f.}$ at least one order of magnitude larger than if such corrections are incorporated;
(3) Fixing $k_m = 0$ and allowing ω to vary gave a final estimate $\omega \sim 0.1-0.2$, which is too low to qualify as a *bona fide* correction-to-scaling exponent; the same happens if one allows $k_m \geq 1$ with a variable ω;
(4) For fixed ω, using $k_m = 1$ reduces the $\chi^2_{\rm d.o.f.}$ by a factor of 2–3 compared with making $k_m = 0$, while no noticeable improvement is forthcoming from allowing $k_m > 1$, again in line with Ref. 13;
(5) For fixed ω between 1 and 2, results for η and $1/\nu$ are in fair accord with point (iii) above; ; also, for this range of ω, $\chi^2_{\rm d.o.f.}$ is minimized, at $\simeq 0.1-0.2$, compared to any alternative combination of fixed and variable parameters described in this paragraph. The coexistence of these facts indicates that, within the assumed scenario of describing corrections to scaling via a single (effective) exponent, the range of ω just quoted is the one that optimizes a universality-consistent picture.

Thus, we kept $j_m = 2$, $k_m = 1$, allowing $1 \lesssim \omega \lesssim 2$ in what follows. Results for $\omega = 1.5$ are shown in Table II.

Since the error bars quoted in the Table only reflect uncertainties intrinsic to the fitting procedure, we now illustrate (see Table III below) the quantitative effects of relaxing some of the assumptions specified above. This is especially important as regards z_c, whose calculated fractional uncertainty is one to two orders of magnitude smaller than those for $1/\nu$, η. Additional checks on the robustness of such narrow error bars are therefore in order.

For instance, considering the T, $\pm J$ case, fixing $\omega = 1,\ 2$ gives respectively $p_c = 0.83611(8)$, $0.83565(6)$, with $\chi^2_{\rm d.o.f.}$ varying by less than 10% against its value for $\omega = 1.5$. Overall, it seems that a realistic error bar should at least include the fitted values of z_c obtained for $\omega = 1$ and 2. Table III shows such estimates, denoted by $z_c^{\rm ave}$, where the associated uncertainties reflect the spread

Table II: TM estimates of critical quantities z_c ($z = p,\ J_0$), $1/\nu$, and η for lattices and coupling distributions [binary ($\pm J$) and Gaussian (G)] specified in column 1. Column 2 gives conjectured values of z_c; quotations from Refs. 4,5,8, and (for T and HC lattices with G distribution) Ref. 15 refer to original conjecture, while all others refer to improved conjecture. All fits used $\omega = 1.5$ (fixed), see Eq. (7) and text.

Type	conj.	$p_c,\ J_{0c}$	$1/\nu$	η	$\chi^2_{\rm d.o.f.}$
SQ, $\pm J$	0.889972[4,5]	0.89061(6)	0.64(2)	0.689(2)	15/116
	0.89079(6)[a]				
SQ, G	1.021770[4,5]	1.0193(3)	0.65(3)	0.680(2)	28/133
	1.021564[15]				
T, $\pm J$	0.835806[8]	0.83583(6)	0.65(2)	0.691(2)	18/119
	0.83597(2)[b]				
HC, $\pm J$	0.932704[8]	0.93297(5)	0.65(1)	0.702(2)	15.5/79
	0.93260(1)[b]				
T, G	0.798174[15]	0.7971(2)	0.66(2)	0.689(2)	17/127
HC, G	1.270615[15]	1.2689(3)	0.64(3)	0.690(2)	11/87

[a]average over four values from improved conjecture, Ref. 15
[b]average over two values from improved conjecture, Ref. 15

Table III: For lattices and coupling distributions [binary ($\pm J$) and Gaussian (G)] specified in column 1, column 2 gives critical quantities $z_c^{\rm ave}$ ($z = p,\ J_0$), averaged over values from fits for $\omega = 1$ and 2 (see text). Coefficients b_0, b_1 (see Eq. (7)) fitted for $\omega = 1.5$; the index (0) for the last two columns denotes quantities obtained from fits where corrections to scaling were neglected.

Type	$z_c^{\rm ave}$	b_0	b_1	$z_c^{(0)}$	$\chi^{2\,(0)}_{\rm d.o.f.}$
SQ, $\pm J$	0.89065(20)	−0.126(3)	−3.7(4)	0.8898(1)	416/118
SQ, G	1.0193(4)	0.009(3)	−0.35(18)	1.0195(1)	30/135
T, $\pm J$	0.83588(23)	−0.145(3)	−1.4(4)	0.8348(1)	520/121
HC, $\pm J$	0.93300(15)	−0.142(3)	−2.5(4)	0.9322(1)	499/81
T, G	0.7972(6)	−0.106(3)	−0.31(18)	0.7948(3)	163/129
HC, G	1.2691(10)	−0.152(4)	−0.64(15)	1.2635(9)	325/89

between these extreme values (their own intrinsic uncertainties generally being somewhat smaller, see above and Table II). A remarkable exception is the SQ, G case, for which the estimate of J_{0c} is virtually unchanged as ω varies in the range described. This instance is also an exception in that the amplitude of the correction term b_0 (column 3 of the Table) is much smaller than for all other cases; consequently, neither J_{0c} nor the χ^2 (resp. columns 4 and 5) change appreciably when corrections to scaling are ignored. The latter is not true for any of the other cases studied.

Our assessment of the estimates quoted in Table III for the location of the NP is as follows.

For SQ, $\pm J$ our results are in agreement with the improved conjecture[15], and with numerical data from Refs. 10,11,12,13. For T, $\pm J$ our range of estimates is roughly consistent with the conjecture, both in its original[8] and improved[15] versions. It is also at the upper limit of the early estimate $p_c = 0.8355(5)$[16].

For all remaining cases, our numerical data indicate that the conjecture fails to hold, albeit by rather small amounts, 0.2% at most. For the $\pm J$ distribution, our results for both SQ and HC indicate that the conjectured position of the NP lies in the paramagnetic phase (for SQ, this is true only for the original conjecture). On the other hand, for the Gaussian distribution and all three lattices, according to our estimates the conjecture places the NP slightly inside the ordered phase.

For HC, $\pm J$ the result in Table III is again at the upper end of the range given in Ref. 16, $p_c = 0.9325(5)$. Note also that our estimate for SQ, G lies farther from the conjecture than the numerical value given in Ref. 11, namely $J_{0c} = 1.02098(4)$ (thus, this latter also places the conjectured location of the NP inside the ordered phase). The above estimates of p_c and J_{0c} for T and HC lattices, when plugged into Eq. (4), using Eq. (3) and its counterpart for Gaussian distributions[15], result in:

$$H(p_{1c}) + H(p_{2c}) = 0.9986(12) \qquad (\pm J)\ ;$$
$$H(J_{0c1}) + H(J_{0c2}) = 1.0014(10) \quad \text{(Gaussian)}\ , \qquad (8)$$

both narrowly missing the conjecture of Eq. (4).

As regards the correlation-length exponent and the critical amplitude η, we found that, for each lattice and coupling distribution, the error bars quoted in Table II are wide enough to accommodate the variations in central estimates, both when one sweeps ω between 1 and 2 as above, and when z_c is varied between the limits established in Table III. No evidence emerges from the data which justifies challenging our earlier assumption of universality. From unweighted averages over the respective columns of Table II, we quote $\nu = 1.53(4)$, $\eta = 0.690(6)$. These are to be compared to the recent results $\nu = 1.50(3)$ (SQ, Ref. 12), 1.48(3) (SQ, Ref. 11), 1.49(2) (T and HC, Ref. 16), 1.527(35) (SQ, Ref. 13), all for $\pm J$ distributions; see also $\nu = 1.50(3)$ (SQ, Ref.11), Gaussian. For the critical amplitude, we recall (all for $\pm J$): $\eta = 0.691(2)$[10,12,24] (SQ); 0.674(11) (T), 0.678(15) (HC), both from Ref. 16 .

The overall quality of our scaling plots is illustrated in Figures 1 and 2. We chose to display data for T and HC, Gaussian distributions, because for these there are fewer data available in the literature. As the last column of Table II shows, the $\chi^2_{\rm d.o.f.}$ remains very much in the same neighborhood for all cases studied.

III. CENTRAL CHARGE

We used the free-energy data generated in Section II also to estimate the conformal anomaly, or central charge c, at the NP. This is evaluated via the finite-size scaling of the free energy on a strip with periodic boundary con-

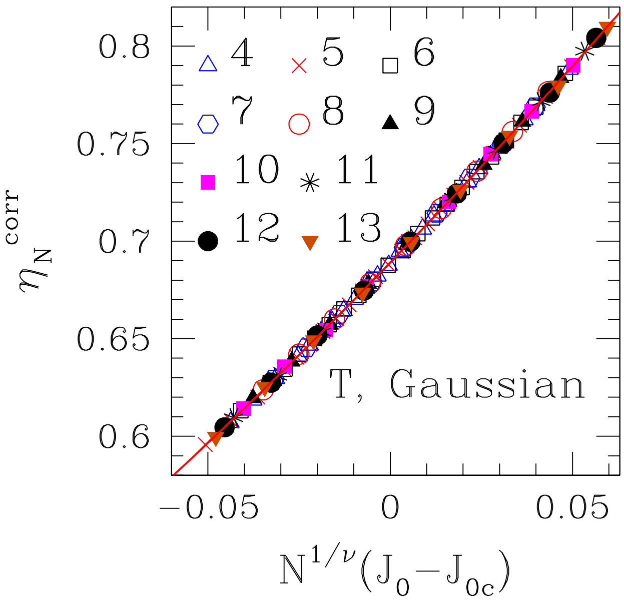

Figure 1: (Color online) Triangular lattice, Gaussian coupling distribution: scaling plot of domain-wall free energies, subtracting corrections to scaling: $\eta_N^{\rm corr} = \eta_N - N^{-\omega} g(N^{1/\nu}(J_0 - J_{0c}))$ [see Eq. (6)], against the scaling variable $N^{1/\nu}(J_0 - J_{0c})$. Error bars are smaller than symbol sizes. The full line is a quadratic fit to corrected data.

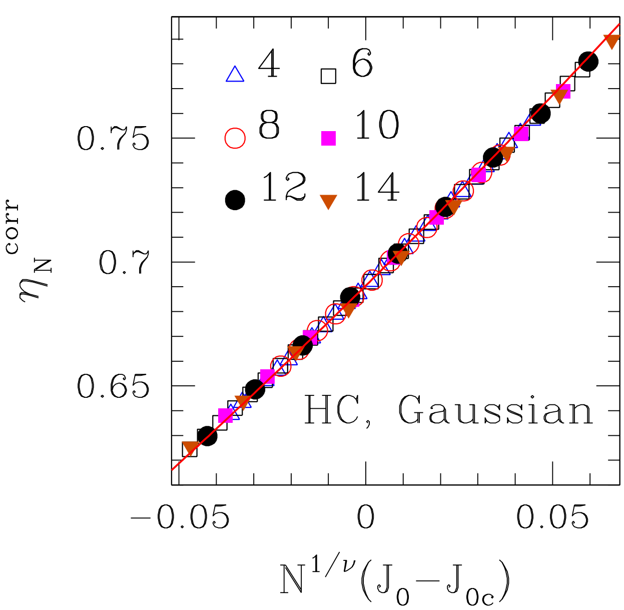

Figure 2: (Color online) Honeycomb lattice, Gaussian coupling distribution: scaling plot of domain-wall free energies, subtracting corrections to scaling: $\eta_N^{\rm corr} = \eta_N - N^{-\omega} g(N^{1/\nu}(J_0 - J_{0c}))$ [see Eq. (6)], against the scaling variable $N^{1/\nu}(J_0 - J_{0c})$. Error bars are smaller than symbol sizes. The full line is a quadratic fit to corrected data.

Table IV: Conformal anomaly c and non-universal higher-order coefficent d, from fits of critical free-energy data to Eq. (9). Last column gives fitted values of c under the asssumption that $d \equiv 0$ in Eq. (9).

Type	c	d	$c\,[d \equiv 0]$
SQ, $\pm J$	0.463(3)	0.13(1)	0.478(2)
SQ, G	0.461(4)	0.14(3)	0.476(2)
T, $\pm J$	0.459(3)	0.01(1)	0.461(1)
HC, $\pm J$	0.457(5)	0.02(2)	0.462(2)
T, G	0.454(4)	0.06(3)	0.461(1)
HC, G	0.468(15)	−0.05(6)	0.459(5)

ditions across[30],

$$f(T_c,N) = f(T_c,\infty) + \frac{\pi c}{6N^2} + \frac{d}{N^4} + \mathcal{O}\left(\frac{1}{N^6}\right) \qquad (9)$$

where $f(T_c,\infty) = \lim_{L\to\infty} f(T_c,L)$ is a regular term which corresponds to the bulk system free energy. For disordered systems, Eq. (9) is expected to hold when the configurationally-averaged free energy is considered, with c taking the meaning of an *effective* conformal anomaly[11,31,32] .

By writing only even powers of N^{-1} in Eq. (9), it is assumed that only analytic corrections come up[33]. While this is true, e.g., for pure Ising systems, a counterexample is the three-state Potts ferromagnet for which free-energy corrections in $N^{-2\omega_0}$, $N^{-3\omega_0}$ …, $\omega_0 = 4/5$, are present[28,34]. Although not much is known about the operator structure at the NP, existing central charge estimates in this case have been derived via Eq. (9) so far with fairly consistent results, namely c=0.464(4)[10,11], 0.46(1)[35]. We shall return to this point at the end of this Section.

We have evaluated free energies at the predicted locations of the NP given in Table III, both at the central estimates and at either end of the respective error bars. We found that such values can be calculated with sufficient accuracy, via interpolation of those already computed at the sets of equally-spaced points used originally in Section II. Results for the central charge are displayed in Table IV, where error bars for all quantities mostly reflect uncertainties intrinsic to the fitting procedure itself, as our estimates are rather stable along the predicted intervals of location of the NP. Indeed, it is expected[11] that at criticality the calculated conformal anomaly passes through a maximum as a function of position along the NL.

In line with earlier findings[35], one sees that for SQ and both coupling distributions, ignoring the fourth-order term in Eq. (9) shifts the final estimate of c by some $4-5$ error bars, away from the expected universal value ~ 0.46. On the other hand, for T and HC the fitted d is much closer to zero than for SQ; furthermore, for these latter lattices, results obtained fixing $d = 0$ appear generallly more consistent with universality, and with less

Table V: For the zero-field susceptibility and lattices as specified in column 1 (all with Gaussian coupling distributions), columns 2, 3 , 4 give the leading correction-to scaling exponent as fitted, and the corresponding γ/ν and $\chi^2_{\rm d.o.f.}$; columns 5, 6 give the two latter quantities, now taken by keeping ω fixed during the fitting procedure, and averaging over resulting values for $\omega = 1$ and 2 (see text).

Type	$\omega^{\rm fit}$	γ/ν	$\chi^2_{\rm d.o.f.}$	$(\gamma/\nu)^{\rm ave}$	$\chi^{2\,{\rm ave}}_{\rm d.o.f.}$
SQ, G	1.3(3)	1.79(2)	0.127	1.793(6)	0.129
T, G	0.7(3)	1.81(1)	0.118	1.814(6)	0.128
HC, G	0.4(6)	1.79(4)	0.17	1.804(7)	0.18

spread, than those found with d kept as a free parameter. Overall, we interpret the above results as indicating that: (i) there is no evidence for universality breakdown as regards the conformal anomaly; taking this as true, (ii) there appears to be no unusual (non-analytic) free-energy scaling correction $N^{-\omega_0}$ with $2 < \omega_0 < 4$; and (iii) it is possible that the fourth-order term is $d \equiv 0$ for T and HC, similarly to the case of pure Ising systems[28].

IV. UNIFORM SUSCEPTIBILITIES

We calculated uniform zero-field susceptibilities along the NL, for SQ, T and HC lattices and only for Gaussian distributions, as done in previous investigations for $\pm J$[16,36]. For the finite differences used in numerical differentiation, we used a field step $\delta h = 10^{-4}$ in units of T. We swept the same respective intervals of J_0 quoted in Table I.

Finite-size scaling arguments[21] suggest a form

$$\chi_N = N^{\gamma/\nu}\, f[\, N^{1/\nu}(J_0 - J_{0c})\,] \;, \qquad (10)$$

where χ_N is the finite-size susceptibility, and γ is the susceptibility exponent. In order to reduce the number of fitting parameters, we kept $1/\nu$ and J_{0c} fixed at their central estimates obtained in Sec. II, and allowed γ/ν to vary. Again, we took corrections to scaling into account. Following Ref. 13, we write:

$$\ln\chi = \frac{\gamma}{\nu}\ln N + \sum_{j=1}^{j_m} a_j\,(J_0 - J_{0c})^j\, N^{j/\nu} + \\ + N^{-\omega}\sum_{k=0}^{k_m} b_k\,(J_0 - J_{0c})^k\, N^{k/\nu} \;. \qquad (11)$$

Similarly to Section II above, we found that choosing $j_m = 2$, $k_m = 1$ enables one to obtain good fits to numerical data, with $\chi^2_{\rm d.o.f.} \simeq 01. - 0.2$. The consequences of keeping ω as a free parameter or, on the other hand, fixing its value during the fitting procedure, can be seen in Table V. While the fitted value of ω for SQ looks acceptable, the same cannot be said of that for HC, as the associated error bar allows even slightly negative values (the result for T being half-way between the other two). Also, by keeping ω as a free parameter, one gets an error bar for γ/ν that is at least twice that obtained if ω is kept fixed between 1 and 2, without any noticeable improvement in the $\chi^2_{\rm d.o.f.}$. On the other hand, using fixed ω above this latter range results in a slow but steady loss of quality: for example, for the T lattice, $\omega = 4$ gives $\chi^2_{\rm d.o.f.} \simeq 0.2$. Thus, although the idea of allowing ω to vary freely seems, in principle, the correct thing to do, the results in this particular case do not appear to be obviously more reliable than those averaged for fixed ω between 1 and 2. We then decided to use these latter as our main reference. Taking an unweighted average over the three estimates for $(\gamma/\nu)^{\rm ave}$ gives the final value $\gamma/\nu = 1.804(16)$. This is to be compared to the following (all for $\pm J$ distributions): 1.80(2)[36] [SQ]; 1.795(20) [T] and 1.80(4) [HC], both from Ref. 16; 1.80–1.82[11] [SQ].

V. CORRELATION FUNCTIONS

Our study of correlation functions is based on previous work for SQ[37], T and HC lattices[16] ($\pm J$ only). In this Section, we specialize to the Gaussian distribution, for T and HC lattices only. On the NL, the moments of the PDF for the correlation function between Ising spins σ_i, σ_j are equal two by two[1,2,38,39]:

$$[\,\langle\sigma_i\sigma_j\rangle^{2\ell-1}] = [\,\langle\sigma_i\sigma_j\rangle^{2\ell}] \;, \qquad (12)$$

where angled brackets indicate thermal average, square brackets stand for configurational averages over disorder, and $\ell = 1, 2, \dots$.

At the NP, conformal invariance[40] is expected to hold, provided suitable averages over disorder are considered[10,12,16,24,37,41,42,43]. On a strip of width L of a square lattice, with periodic boundary conditions across, the disorder-averaged k-th moment of the correlation function PDF between spins located respectively at the origin and at (x, y) behaves at criticality as:

$$[\,\langle\sigma_i\sigma_j\rangle^k] \sim z^{-\eta_k} \;,\; z \equiv [\,\sinh^2(\pi x/L) + \sin^2(\pi y/L)\,]^{1/2} \;. \qquad (13)$$

For the T and HC lattices, the same is true, provided that the actual, i.e., geometric site coordinates along the strip are used in Eq. (13). Details are given in Ref. 16. Note that Eq. (12) implies $\eta_{2\ell-1} = \eta_{2\ell}$.

As in earlier work[16,37], we concentrate on short-distance correlations, i.e., where the argument z is strongly influenced by y. Such a setup is especially convenient in order to probe the angular dependence predicted in Eq. (13), which constitutes a rather stringent test of conformal invariance properties.

Following Refs. 10,16,37, we extract the decay-of-correlations exponents η_k, via least-squares fits of our data to the form $m_k \sim z^{-\eta_k}$. We also consider the exponent η_0 which characterizes the zeroth-order moment of the correlation-function distribution[35], i.e. it gives the

typical, or most probable, value of this quantity (see, e.g., Ref. 43 and references therein). One has, in the bulk,

$$G_0(R) \equiv \exp\left[\ln\langle\sigma_0\sigma_R\rangle\right]_{\rm av} \sim R^{-\eta_0}\ . \qquad (14)$$

Calculations on strips of the $\pm J$ SQ lattice, at the early conjectured location of the NP[4], gave the estimate $\eta_0 = 0.194(1)$[35] .

As seen in earlier work[37], for strip widths $N = 10$ or thereabouts, finite-width effects are already mostly subsumed in the explicit L (i.e., N) dependence of Eq. (13). However, some detectable (albeit tiny) variations in the calculated values of averaged moments of the correlation function PDF may still persist upon varying N. These are of course minimized at the critical point where the bulk correlation length diverges. We have calculated correlation functions for $N \leq 12$, for values of J_0 within the error bars given for the location of the NP in Table III . We have seen that along these intervals of J_0, the trend followed by such averaged moments against $N-$ variation is as follows: for T, it cannot be distinguished from stability within error bars, while for HC it is slightly downward (of the order of one error bar from $N = 10$ to $N = 12$). For fixed N and J_0, one error bar associated to intrinsic fluctuations is $\lesssim 1\%$.

Table VI gives numerical results of the fits for $k = 0$, and odd $k > 1$ (we have also calculated even moments for $k \geq 2$ and checked that Eq. (12) holds). One sees that T and HC estimates are quite consistent with each other for all k. On the other hand, for $k = 0$, 1, and 3 they fall slightly below their existing counterparts, given in columns 4, 5, and 6 of the Table. For $k = 5$ and 7, as a consequence of generally wider error bars, all estimates are broadly compatible with one another.

Physically, obtaining (via least-squares fits) a smaller [larger] than expected value for the decay-of-correlations exponent would indicate that it is being evaluated inside the ordered [paramagnetic] phase, instead of right at the critical point.

Applying these ideas to the $k = 0$ case, we recall that the result of Ref. 35 for SQ, $\pm J$ was calculated at the originally conjectured position of the NP[4]. By now, it seems well established that this point is in the disordered phase (see Table II). Therefore, the value of Ref. 35 should be taken as an upper bound, which is obeyed by our present estimates.

For $k = 1$ and 3, one might use the same argument as above to argue that the result of Ref. 37 is too large, as it was calculated at the same point as that of Ref. 35. On the other hand, this cannot be said of the additional estimates quoted in the Table, all of which are also larger than ours (though in some cases the respective error bars overlap, or at least touch each other). Using the reasoning described above, one would infer that for T and HC with Gaussian distribution, the ranges of locations for the NP given in Table III are in fact both inside the ordered phase. Since these latter, in their turn, put the conjectured NP position also inside the ordered phase, the final conclusion would be that the actual location of the NP differs from the conjecture by an amount larger than predicted by domain-wall (DW) scaling: $J_{0c}^{\rm real} < J_{0c}^{\rm DW} < J_{0c}^{\rm conj.}$. The slight downward trend against increasing N, reported above for HC, would be consistent with this scenario. However, we have not seen a similar trend for T.

Table VI: Estimates of exponents η_k, from least-squares fits of averaged moments of correlation-function distributions to the form $m_k \sim z^{-\eta_k}$, for z defined in Eq. (13). For columns 2 and 3 (this work), central estimates and error bars reflect averages between results for $N = 10$ and 12, as well as variations from scanning J_0 along the error bars for locations of NP given in Table III. Columns 4, 5, 6 quote existing data for comparison. For SQ, all results are for $\pm J$ coupling distribution, unless otherwise noted.

k	T (G)	HC (G)	T $(\pm J)$[16]	HC $(\pm J)$[16]	SQ
0	0.185(3)	0.184(3)	–	–	0.194(1)[35]
1	0.178(2)	0.178(2)	0.181(1)	0.181(1)	0.1854(17)[37]
					0.1854(19)[10]
					0.183(3)[12]
					0.1848(3)[11]
					0.1818(2) [G][11]
					0.180(5)[13]
3	0.250(2)	0.252(2)	0.251(1)	0.252(1)	0.2556(20)[37]
					0.2561(26)[10]
					0.253(3)[12]
					0.2552(9)[11]
					0.2559(2) [G][11]
5	0.296(2)	0.300(5)	0.297(2)	0.296(2)	0.300(2)[37]
					0.3015(30)[10]
					0.3004(13)[11]
					0.3041(2) [G][11]
7	0.331(4)	0.336(6)	0.330(2)	0.329(3)	0.334(3)[37]
					0.3354(34)[10]
					0.3341(16)[11]
					0.3402(2) [G][11]

One should note also that all the discrepancies remarked upon are rather small: the single worst case, as regards central estimates, is that of the present result $\eta_1 = 0.178$ against $\eta_1 = 0.1854$[10,37], amounting to 4%, or $\simeq 3.5$ times the respective error bar. Given that the quoted values (especially those for the associated uncertainties) are likely to depend on details of the respective fitting procedures, the resulting picture looks mixed.

In conclusion, existing evidence does not seem strong enough to state that our estimates from Sec. II for the location of the NP on T and HC lattices (Gaussian distribution) are definitely inside the ordered phase.

VI. DISCUSSION AND CONCLUSIONS

We have used domain-wall scaling techniques in Sec. II to determine the location of the Nishimori point of Ising spin glasses. In the analysis of our data we allowed for the existence of corrections to scaling, see Eqs. (6) and (7).

Results for the SQ lattice, $\pm J$ distribution, show that such corrections play a crucial role in the finite-size scaling of domain-wall energies. Indeed, when they are taken into account, the estimated position of the NP is $p_c = 0.89065(20)$, in excelent agreement with recent and very accurate numerical work[11,13]. One can see from the two last columns of Table III that, if corrections to scaling are ignored, the value of p_c which minimizes the $\chi^2_{\rm d.o.f.}$ (though at a level ~ 30 times that obtained when corrections are incorporated) is instead $p_c = 0.8898(1)$, very close to the original conjecture and incompatible with the above-mentioned body of numerical evidence. In retrospect, one sees that the domain-wall scaling result of Ref. 16 for this case, namely $p_c = 0.8900(5)$, essentially suffers from the effect of ignoring corrections to scaling (though even so it still picks out the correct exponent, $1/\nu = 1.45(8)$[16]).

Going over to SQ, Gaussian, domain-wall scaling for strips of widths $4 \leq N \leq 14$ sites gives $J_{0c} = 1.0193(4)$, lower than both the conjecture (original and improved) and the result of Ref. 11, $J_{0c} = 1.02098(4)$). In that Reference, the mapping of the spin problem to a network model (described in Ref. 12) enabled the authors to reach significantly larger lattice sizes than here. The result just quoted was obtained by extrapolation of $11 \leq N \leq 24$ crossing-point data, without explicit account of corrections to scaling (which, as those authors show, do produce a trend reversal around $N = 8$, and are expected to have negligible effect for the large widths used in the extrapolation). It may be that our own data fail to incorporate an underlying trend which only comes about for larger systems. Nevertheless, the stability of our results for this particular case is remarkable, as pointed out in the initial discussion of Table III.

Our results for T and HC, $\pm J$ distribution, are marginally compatible with, but more accurate than, the earlier ones of Ref. 16; though for T they are also broadly compatible with the conjecture in both original and improved versions, for HC our estimate in Table III lies at least two error bars away from the conjecture.

For T and HC, Gaussian distribution, in both cases the discrepancy between our results and the conjecture is again of the order of two error bars.

Consequently, as shown in Eq. (8), we predict the duality-based conjecture of Eq. (4) to be narrowly missed, for both $\pm J$ and Gaussian cases, though on opposite sides of the hypothesized equality.

As regards the exponent ν and critical amplitude η (see Eq. (5)) which are also estimated via domain-wall scaling, we have found no evidence of nonuniversal, lattice– or disorder distribution– dependent, behavior. Therefore, from unweighted averages over all six cases studied, we quote $\nu = 1.53(4)$, $\eta = 0.690(6)$. Both are in very good agreement with existing numerical results (see the end of Sec. II for detailed comparisons).

The conformal anomaly values calculated in Sec. III are in good agreement among themselves and with previous estimates[10,11,35]. Our fits for the non-universal coefficient d of the fourth-order correction to the critical free energy suggest that $d \equiv 0$ for T and HC lattices (while definitely $d \neq 0$ for SQ). This would be similar to the lattice-dependent structure of corrections for pure Ising systems[28]. An unweighted average of values from Table IV (using results of fits with $d \neq 0$ for SQ, and with $d \equiv 0$ for T and HC) gives $c = 0.461(5)$.

In Sec. IV we evaluated uniform zero–field susceptibilities, by direct numerical differentiation of the free energy against external field. Only Gaussian distributions were considered, for SQ, T, and HC. Though our results show some lattice-dependent spread, the error bars for $(\gamma/\nu)^{\rm ave}$ still overlap in pairs. It is known that susceptibility calculations are prone to larger fluctuations than, e.g., domain-wall energy ones[11]. In the Monte Carlo simulations of Ref. 13, this effect was reduced by considering the quantity χ/ξ^2 (where ξ is the finite-size correlation length), which behaves more smoothly than χ on its own. Our final estimate (averaged over results for all three lattices), $\gamma/\nu = 1.804(16)$, compares favorably (albeit somewhat close to marginally) with the corresponding one from Ref. 13, $\gamma/\nu = 1.820(5)$.

Finally, in Sec. V we applied conformal-invariance concepts to the statistics of spin-spin correlation functions, extracting the associated multifractal scaling exponents[10,11,12,24]. We only examined T and HC lattices, for Gaussian coupling distributions. The overall picture summarized in Table VI points towards universality of the exponents $\{\eta_k\}$, though some small discrepancies remain. The case $k = 1$ is especially relevant, on account of its connection with the uniform susceptibility via the scaling relation $\gamma/\nu = 2 - \eta_1$. While our result $\eta_1 = 0.178(2)$ is somewhat lower than existing data from direct calculations of correlation functions, it gives $\gamma/\nu = 1.822(2)$ when inserted in the scaling relation. This agrees very well with the above-quoted estimate[13], $\gamma/\nu = 1.820(5)$.

In summary, we have produced estimates of the location of the NP on SQ, T and HC lattices, and for $\pm J$ and Gaussian coupling distributions. Though these are consistent with existing conjectures for SQ and T (both $\pm J$), they appear to exclude the respective conjectured values for the remaining cases. However, the discrepancies are very small, amounting to 0.2% in the worst case (SQ, Gaussian). Furthermore, we have assessed several critical quantities (amplitudes and exponents), and found an overall picture consistent with universality as regards lattice structure and disorder distribution.

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